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Suggested answers

NA0167 Exam, Jan. 2015.

1. Consider the neoclassical growth model with zero population growth, and add the need for a resource input in the production function. The flow of the resource input is denoted R, there is a total stock of the resource S, and there is no substitute for the resource. There is exogenous labour-augmenting technological progress at a constant rate. The discount rate (interest rate) is exogenously fixed at ρ.

$$\begin{aligned} Y &= (AL)^{1-\alpha-\beta}K^{\alpha}R^{\beta}, \\ \dot{A}/A &= g_A, \\ \dot{K} &= sY - \delta K, \\ S &\geq \int_0^{\infty} R_t \mathrm{d}t. \end{aligned}$$

   (a) Assume that there exists a balanced growth path (b.g.p.) on which all variables grow at constant rates. Find the growth rate of Y on this b.g.p. in terms of parameters and $\dot{R}/R$.

   (b) Consider a competitive final-good producer who makes profits $\pi = (AL)^{1-\alpha-\beta}K^{\alpha}R^{\beta} - w_l L - w_k K - w_r R$. (The price of Y is normalized to 1.) Take the first-order condition in R to find an expression for w_r in terms of Y and R.

   (c) Use your previous answers to find an expression for the growth rate of w_r in terms of parameters and $\dot{R}/R$.

   (d) Imagine that you own a small fraction of the total resource stock S, and that the resource is traded on a perfect market (there are many other small suppliers). How do you decide when to sell your resource stock? Given that the other suppliers reason in the same way as you, what must be the growth rate of w_r? Now use your answer to part (c) to find the growth rate of R.

   (e) Discuss briefly the extent to which this model might give a useful—although highly simplified—picture of economic growth and resource use in the long run.

(a) We can immediately obtain that

$$\dot{Y}/Y = (1-\alpha-\beta)g_A + \alpha\dot{K}/K + \beta\dot{R}/R.$$

But we know from the capital accumulation equation that $\dot{Y}/Y = \dot{K}/K$ on a b.g.p., so

$$\dot{Y}/Y = [1 - \beta/(1-\alpha)]g_A + \beta/(1-\alpha)\dot{R}/R.$$

(b) The f.o.c. yields directly that

$$w_r = \beta Y/R.$$

(c) From the answer for w_r we obtain directly that

$$\dot{w}_r/w_r = \dot{Y}/Y - \dot{R}/R,$$

and hence (using the answer to (a)) we have

$$\dot{w}_r/w_r = [1 - \beta/(1-\alpha)](g_A - \dot{R}/R).$$

(d) If the price were rising at a faster rate than money at the bank then I would keep the resource, whereas if it were rising more slowly then I would sell. If all other suppliers reason the same way then w_r must grow at the interest rate, i.e. $\dot{w}_r/w_r = \rho$. Hence

$$\dot{R}/R = g_A - \rho[1 - \beta/(1-\alpha)]^{-1} = g_A - \frac{1-\alpha}{1-\alpha-\beta}\rho.$$

(e) In my opinion the picture painted by the model of economic growth and resource use in the long run is misleading and not at all useful. The key reason is that the model of resource stocks and resource extraction is far too simple. A model is needed in which resources are costly to extract, and where in addition resource stocks are inhomogeneous so that the cost of extraction tends to increase as cumulative extraction increases. Given such a model we can explain current observations of rising extraction and constant prices at the same time as making predictions about future extraction rates and prices.

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2. Consider an economy with a constant population and a constant interest rate r, and the following aggregate production function:

$$y = (al_y)^{1-\alpha}x^\alpha.$$

Here α is a parameter between 0 and 1, a is labour productivity, and l_y and x are the quantities of labour and resources used in production. The resource flow x is given by the following extraction function:

$$x = l_x a/b.$$

Labour inputs in extraction are l_x, the productivity of that labour is a (as above), and b is an inverse productivity factor representing the difficulty of extracting the resource: the *depth* of the marginal resource. The productivity index a grows exogenously, and total labour L is fixed:

$$\dot{a}/a = \theta_a$$
$$L = l_x + l_y.$$

Finally, we assume that all markets are perfect, and we have a unit continuum of resource owners each with identical inhomogeneous endowments.

(a) Find an expression for the resource price p in terms of y and x, by taking the first-order condition on a final-good producer's profit function.

(b) Assume a b.g.p. on which quantities of labour are constant and depth b grows at a constant rate. Find expressions for the growth rate of x, p, and y on such a b.g.p.

(c) Assume a primitive economy in which resource extraction is just beginning. What can we say about b? Characterize the b.g.p.! What happens over time?

(d) Assume a 'mature' b.g.p. on which b grows at a constant strictly positive rate. For concreteness assume that $\dot{b}/b = \theta_a$ on this path. Characterize the b.g.p.!

(e) Finally, characterize the development of the economy if the resource is close to exhaustion, and there are no substitutes.

(a) The first-order condition yields

$$p = \alpha y/x.$$

(b) On a b.g.p., when labour allocation is fixed, the following growth rates apply:

$$\begin{aligned}
\dot{x}/x &= \theta_a - \dot{b}/b; \\
\dot{y}/y &= (1-\alpha)\theta_a + \alpha\dot{x}/x \\
&= \theta_a - \alpha\dot{b}/b; \\
\dot{p}/p &= \dot{y}/y - \dot{x}/x = (1-\alpha)\dot{b}/b.
\end{aligned}$$

(c) When extraction is just beginning then depth b must be very nearly constant, so we can approximate $\dot{b}/b = 0$. So we have

$$\begin{aligned}
\dot{x}/x &= \dot{y}/y = \theta_a; \\
\dot{p}/p &= 0.
\end{aligned}$$

Resource extraction tracks growth in GDP and the resource price is constant! Over time, since the extraction rate rises, b starts to increase significantly and the price starts to rise.

(d) Now we have

$$\begin{aligned}
\dot{x}/x &= 0; \\
\dot{p}/p &= \dot{y}/y; \\
\dot{y}/y &= (1-\alpha)\theta_a.
\end{aligned}$$

So the resource extraction rate is constant, the resource price tracks GDP, and GDP growth is slightly slower.

(e) When the resource is close to exhaustion then the depth again becomes constant, and extraction approaches zero. A extraction approaches zero the price rises rapidly, but not due to extraction costs, rather due to scarcity. In the limit the scarcity rent dominates completely and the price rises at rate ρ as in the simple Hotelling model.

3. > [A]s the earth's supply of particular natural resources nears exhaustion, and as natural resources become more and more valuable, the motive to economize those natural resources should become as strong as the motive to economize labor. The productivity of resources should rise faster than now—it is hard to imagine otherwise.
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> [Solow, *Is the end of the world at hand?*, Challenge, 1973, p47.]

   (a) Over the last 300 years the price of energy has been falling compared to the price of labour.

      i. Explain why, in theory, this might lead labour-augmenting knowledge to grow faster than energy-augmenting knowledge.

      ii. Discuss evidence.

   (b) Over the next 50 years the price of energy may well rise relative to the price of labour. Will this lead to rapid increases in the efficiency of energy use in sectors such as lighting and transport? Discuss theory and evidence.

(a) i. Here you should explain why, because labour and energy are complementary, the fall in the price of energy might be expected to lead to a fall in the factor share of energy. Such a fall will lead to a fall in investment in energy-augmenting knowledge. This may cause energy-augmenting knowledge to fall relative to labour-augmenting knowledge, which would tend to push the factor share of energy back up since it effectively makes energy scarcer. So this could explain why the factor share of energy has stayed rather constant even though the price of energy has fallen relative to labour.

   ii. The evidence suggests that energy-augmenting knowledge has grown at least as fast as labour-augmenting knowledge. Evidence we discussed in the course concerns lighting and motive power from combustion of fossil fuels. More generally, there are myriad uses to which we can put energy today compared to 300 years ago. Each of these uses implies a completely new stock of (product-specific) 'energy-augmenting knowledge'.

(b) In the simple one-sector model with independent knowledge stocks, a rise in the price of energy should drive a rise in energy-augmenting knowledge. Conversely, when prices are constant such knowledge should fail to grow. But the evidence cited in part (a) leads us to reject this model.

   Progress in energy-efficiency is not a stationary function of investment. In sectors such as lighting and transport there are well defined limits to energy efficiency: for instance, there is a limit to the amount of light (lumens) that can be generated from a given energy input, and there is a limit to the amount of motive power that can be generated from a given energy input. Furthermore, we are approaching these limits; LED lights and the latest internal combustion engines can be improved upon, but their efficiency cannot be doubled and doubled again. In the case of lighting, Fouquet claims that lighting efficiency increased by a factor of 1000 in the UK between 1800 and 2000. But the latest LED lights are at close to 50 percent of maximum efficiency, so only a factor of 2 remains available for the future.

   On the other hand, note that efficiency improvements in some other sectors—such as domestic heating—may well be limitless, and we may be able to approach a long-run situation in which homes can be held at the desired temperature with zero external energy inputs.

4. Assume an economy with competitive markets in which total aggregate production is a function of labour-intensive and energy-intensive production, as follows:

$$Y = Y_1^{\alpha} Y_2^{1-\alpha}.$$

The labour-intensive good is produced according to the following production function:

$$Y_1 = a_l L,$$

where a_l is labour-augmenting knowledge and L is labour, which is fixed. The energy-intensive good is produced according to the following production function

$$Y_2 = a_r R,$$

where a_r is energy-augmenting knowledge, and R is the energy flow. The price of energy, w_r, is fixed in relation to the wage w_l:

$$w_r = \psi w_l,$$

where ψ is a positive parameter. Any amount of energy R can be supplied at this price.

(a) i. Find the relative shares in total product of Y_1 and Y_2. (That is, find $p_1 Y_1/(p_2 Y_2)$, where p_1 and p_2 are the prices of the two goods Y_1 and Y_2.)

ii. Consider unit production costs in order to find p_1 and p_2 in terms of w_l, ψ, a_l, and a_r.

iii. Find total energy use R for a given state of the economy. (This is, when L, a_l, a_r, and ψ are all fixed and known.)

iv. Assume that a regulator wants to reduce R, and that she can either boost ψ (and hence w_r) through a tax, or a_r through a research subsidy. Explain which option she should choose in this economy.

(b) Discuss to what extent the above model is relevant to real economies in which the energy share of the most energy-intensive products is typically only about 15 or 20 percent, rather than 100 percent as in the model.

(a) i. Take first-order conditions on the aggregate producer's profit function to show that

$$p_1 Y_1/(p_2 Y_2) = \alpha/(1-\alpha).$$

ii. The total cost of making quantity Y_1 of good 1 is $w_l L$. Therefore the unit cost is $w_l L/Y_1$, hence

$$p_1 = w_l/a_l,$$

and (by symmetry)

$$p_2 = w_r/a_r = \psi w_l/a_r.$$

iii. We have

$$p_1 Y_1/(p_2 Y_2) = \alpha/(1-\alpha) = \frac{w_l/a_l}{\psi w_l/a_r}\frac{Y_1}{Y_2}.$$

Simplify, and then substitute in the production functions for Y_1 and Y_2 to yield

$$\frac{\alpha}{1-\alpha} = \frac{a_r}{\psi a_l} \cdot \frac{a_l L}{a_r R},$$

and hence

$$R = \frac{1}{\psi}\frac{1-\alpha}{\alpha}L.$$

iv. Boosting ψ seems like a good strategy (the elasticity of demand for R w.r.t. increases in ψ is -1), but boosting a_r won't help at all. There is 100 percent rebound in this economy!

(b) The above model is not very relevant to real economies (in which the energy share of the most energy-intensive products is typically only about 15 or 20 percent). When the energy share of energy-intensive final goods is relatively low this implies that an increase in the energy-efficiency of the production process of such goods has relatively little effect on their price, implying that (for reasonable levels of substitutability between the alternative final goods) the effect on the quantity demanded is not very great, hence the rebound effect is not very large.